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# Model Order Reduction by Selective Sensitivity

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**Many industrial structures are represented by models with a large number of degrees of freedom, thus making their use complex and costly. Model order reduction alleviates this problem by elaborating lower-dimensional models that satisfy some properties of the refined model. We present a model order reduction procedure based on the concept of selective sensitivity. This reduction method has the property of preserving energies and static or dynamic behaviors of the refined model in the reduced model while preserving the essence of its topology.**

## Nomenclature

$K \in R^{N \times N}$ = stiffness matrix of refined model
$\tilde{K} \in R^{m \times m}$ = stiffness matrix of reduced model
$K_i, M_i \in R^{N \times N}$ = submodel connectivity matrices
$k_i, m_i \in R$ = model parameters
$M \in R^{N \times N}$ = mass matrix of refined model
$\tilde{M} \in R^{m \times m}$ = mass matrix of reduced model
$P \in R^{N \times m}$ = projection matrix
$\Theta$ = matrix of selective vectors
$\theta \in R^N$ = selective vector
$\Psi$ = matrix of deselective vectors
$(\ )^T$ = transpose operator

## I. Introduction

THE design and optimization of industrial components often requires the construction of predictive mathematical models. The presence of complex geometries, materials, and assembly techniques—not to mention modern automated CAD tools—inevitably pushes the engineer to employ progressively finer meshes, which can lead to very high models orders. This situation can seriously handicap the implementation of the model in diverse applications because of the high manipulation costs such as memory requirements and calculation times.

Model reduction strategies in linear elastodynamics reduce the order of the model to a usable size while preserving a certain set of behavior characteristics necessary to ensure the success of the end use. Two broad objectives for model reduction typically can be distinguished. The first objective is the need to calculate the output behavior (eigensolutions or particular solutions) of a model. In this case, it is generally assumed that the only prior information available concerning the model is its associated state matrices. The most common examples of this brand of model reduction are the static or dynamic Guyan condensations.[1] The second objective concerns the need for a relatively low-order dynamic stiffness matrix of the model for use in general design, prediction, and optimization applications, for example, dynamic substructuring, model correction, and approximate reanalysis. Although the Guyan method can be employed directly in this context, methods using additional prior information (e.g., a Ritz subbasis formed of modes or static deformations of the refined model) are commonly employed. Now, classical examples of this approach include Craig–Bampton,[2] MacNeal,[3] and Rubin,[4] to cite just a few.

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In general, both classes of reduction methods retain a restricted set of model degrees of freedom, which may be either physical, generalized, or a mixture of both. The primary difficulty in applying these procedures in practice is to understand which coordinate degrees of freedom (physical and/or generalized) must be retained to ensure a sufficiently precise representation of the refined model behavior. An additional and often overlooked inconvenience is that the topological properties of the refined model are generally lost in the reduced model, that is, the contributions of the design parameters affecting the unretained physical degrees of freedom are generally spread over the entire set of retained coordinates.

A new model reduction strategy is proposed that has the distinctive feature of preserving certain topological properties of the refined model while preserving a defined set of spatial and dynamic characteristics. This procedure is based on the concept of selective sensitivity,[5–8] whose orthogonality properties are used to quantify the natural local clustering of model parameter subgroups induced by the connectivity or proximity properties that are characteristic of linear elastic systems.

The properties of this model reduction procedure are described in Sec. II. A specific implementation of the strategy is outlined in Sec. III, and an illustrative example is presented in Sec. IV. The appendix contains formal definitions and proofs of the relevant theorems.

## II. Model Order Reduction Properties

In this section we define what we mean by a model order reduction, and we discuss in qualitative terms the physical properties that such a transformation should have. The precise mathematical basis of the selectively sensitive model order reduction proposed here is developed in the Appendix.

Consider a conservative linear elastic system whose $N$-dimensional refined model is characterized by the stiffness and mass matrices $K$ and $M$. The $N \times N$ dynamic stiffness matrix is given by

$$Z(\lambda) = K - \lambda M \tag{1}$$

We assume that $Z(\lambda)$ can be expressed as a linear function of the model parameters:

$$Z(\lambda) = \sum_{i=1}^{R_k} k_i K_i - \lambda \sum_{i=1}^{R_m} m_i M_i \tag{2}$$

where the real symmetric matrices $K_i$, $M_i$ depend only on the connectivity of the finite element representation of the structure and $k_i$, $m_i$ are scalar model parameters: local stiffnesses and inertias. Representation (2) is referred to as the submodeling property of the model.

We define the stiffness and mass matrices of a model of reduced dimension in terms of the refined-model matrices as

$$\tilde{K} = P^T K P \qquad \tilde{M} = P^T M P \tag{3}$$

where $P$ has fewer columns than rows. The transformation matrix $P$ is defined by a systematic procedure in Sec. III and its application is illustrated in Sec. IV. This transformation invests the reduced-order system with a particular relationship to the refined model, Eq. (2), which we describe.

The most important property of the selectively sensitive reduced-order model is topological localization, defined rigorously in definition 4 of the appendix. Here we describe its intuitive meaning.

A model order reduction has the property of topological localization if the variations of the reduced stiffness matrix $\tilde{K}$ with respect to the refined-model stiffness parameters $k_i$ are independent in the sense of being topologically nonoverlapping. That is, variation of a stiffness parameter $k_i$ in the refined model causes variation of parameters in the reduced model that are unaffected by any other refined parameter. This establishes a unique correspondence between the physical model parameters of the reduced and refined models. This is important in situations where analyses performed on the reduced model are used to suggest physical properties or design changes for the refined model.

The property of topological localization is achieved mathematically by requiring that the model order reduction satisfy Eq. (A5). In proposition 2, this is proven to hold for the selectively sensitive model order reduction and only for such procedures. What Eq. (A5) states is that the stiffness parameters can be organized into groups in such a way that each group of parameters populates a unique part of the stiffness matrix. Consequently, the property of topological localization means that if one parameter of the refined model is modified, then the physical parameters of the reduced model that are affected by this change are easily localized in the stiffness matrix.

Proposition 3 establishes a relationship between the parameters of the refined model and the static responses of the reduced model. In particular, Eqs. (A21) and (A22) demonstrate that the static responses $y_s$ of the reduced model depend in a simple manner on the selectively sensitized stiffness parameters, which appear in $y_s^S(k_j, f)$. This I/O relation depends explicitly on the property of topological localization and clearly is not characteristic of either the refined model or a general reduction transformation. This property allows a particularly cost-effective procedure for choosing the stiffnesses to obtain desired static responses.

Finally, propositions 4–6 are more general and concern any reduction strategy based on a full rank linear transformation between the refined and the reduced model displacements, as defined in Eq. (3). They demonstrate that the static and eigenbehaviors of the refined model are preserved if and only if the corresponding output vectors can be represented on the Ritz basis $P$.

## III. Model Order Reduction Algorithm

The model order reduction of the high-dimensional refined model to the low-dimensional reduced model by the selective sensitivity procedure can be implemented in various ways. One approach is described in the following steps.

Step 1) Select domains of the refined model that are to be decoupled in the reduced model. Each of these domains $D_j$ $(j = 1, \ldots, p)$ is specified by a set containing the indices of the stiffness parameters in the refined model we would like to reduce to one single parameter or, at most, a few parameters in the reduced model.

Step 2) For each submodel domain $D_j$, find a collection of all $d_j$ linearly independent selective vectors $\theta$ satisfying the property

$$K_i\theta = \begin{cases} 0 & i \notin D_j \\ \text{not } 0 & i \in D_j \end{cases} \tag{4}$$

This is the basic property upon which selective sensitivity is based and the existence of such vectors is an important property of linear elastic systems as described elsewhere.[6,8] For the $j$th domain, we calculate all $d_j$ linearly independent selective vectors for the associated stiffness matrices and store them in a matrix $\tilde{\Theta}^j \in R^{N\times d_j}$. These $p$ matrices are grouped in a single matrix $\tilde{\Theta} = [\tilde{\Theta}^1 \ \ldots \ \tilde{\Theta}^p] \in R^{N\times\sum_j d_j}$.

Step 3) If the columns of $\tilde{\Theta}$ do not span the entire space, which will usually be the case, we extend the basis of selective vectors. We define deselective vectors, which are grouped in the matrix $\tilde{\Psi}$, and $\tilde{\Psi}$ is orthogonal to the stiffness matrices associated to the domains $D_1, \ldots, D_p$:

$$K_i\tilde{\Psi} = 0 \qquad i \in D_j, \quad j = 1, \ldots, p \tag{5}$$

Step 4) The columns of $\tilde{\Theta}$ and $\tilde{\Psi}$ form the initial basis of our model reduction, and they are combined in a single matrix as

$$\tilde{P} = [\tilde{\Theta} \mid \tilde{\Psi}] \tag{6}$$

This matrix $\tilde{P}$ is a full column rank rectangular matrix.

Step 5) We compile a matrix $Y^{(m)}$ whose columns are dynamic or static responses of the refined model, which we want to preserve. Mode shape vectors may be included among the columns of $Y^{(m)}$. We search for the projection matrix $\tilde{C} = [\tilde{C}^T_{\Theta^1} \ \ldots \ \tilde{C}^T_{\Theta^p} \mid \tilde{C}^T_{\Psi}]^T$ of $Y^{(m)}$ on the basis $\tilde{P}$, which minimizes the error $\varepsilon$ in the relation

$$\begin{aligned} Y^{(m)} &= \tilde{P}\tilde{C} + \varepsilon \\ &= \sum_j \tilde{\Theta}^j\tilde{C}_{\Theta^j} + \tilde{\Psi}\tilde{C}_{\Psi} + \varepsilon \end{aligned} \tag{7}$$

Because of the orthogonality property, Eqs. (4) and (5), we see by multiplying on the left by $K_j$ that each submatrix $\tilde{C}_{\Theta^j}$ can be independently calculated from

$$K_jY^{(m)} = K_j\tilde{\Theta}^j\tilde{C}_{\Theta^j} \tag{8}$$

The Moore–Penrose pseudoinverse of $K_j\tilde{\Theta}^j$ multiplied by $K_jY^{(m)}$ gives us a minimum norm estimate for $\tilde{C}_{\Theta^j}$.

Step 6) Singular value decompositions are then applied on each component of the right-hand side of Eq. (7) to find the vectors that are the most important in representing the columns of $Y^{(m)}$. The singular value decomposition of $\tilde{\Theta}^j\tilde{C}_{\Theta^j}$ leads to matrices $U_j$, $\Sigma_j$, and $V_j$, which satisfy

$$\tilde{\Theta}^j\tilde{C}_{\Theta^j} = U_j\Sigma_jV_j^T \tag{9}$$

where $\Sigma_j$ is a diagonal matrix and $U_j$, $V_j$ are unitary matrices. From this decomposition, we select the columns of $U_j$ that are associated with the greatest singular values, and store them in a matrix $\Theta^j$. The same operation is performed to choose the best deselective vectors from the singular value decomposition of $\tilde{\Psi}\tilde{C}_{\Psi}$. These vectors define the matrix $\Psi$.

The choice of the number of vectors to retain depends on a figure of merit based on the relation between the representation error of $Y^{(m)}$ on $\Theta^j$ or $\Psi$ and the truncation error on singular values.[9]

Step 7) All of the vectors in $\Theta^1$ to $\Theta^p$ and $\Psi$ are grouped in a matrix $P = [\Theta \mid \Psi]$, which now defines the projection matrix in the model order reduction by selective sensitivity.

Step 8) Finally, the reduced model is constructed by projecting on $P$ the stiffness and mass matrices of the refined model, as shown in Eq. (3).

## IV. Numerical Application

An application of model order reduction by selective sensitivity has been performed on an academic test case. The structure we consider is a fixed-free beam that has been modeled with 50 bidimensional finite beam elements, as shown in Fig. 1 and longitudinal effects have been neglected.

We would like to obtain a reduced model that preserves the first 10 modes of the refined model. So, the first 10 eigenvectors make up the columns of the matrix $Y^{(m)}$, defined in step 5 of Sec. III. We first outline the eight stages by which the model reduction matrix $P$ is constructed, as described in Sec. III.

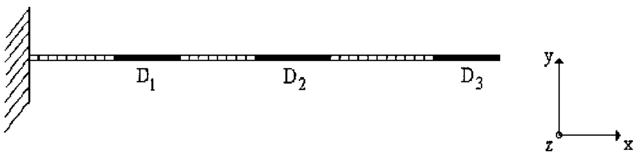



**Fig. 1 Structure BEAM.**

Step 1) The domains we are interested in decoupling, for some particular reasons, are the domains $D_1$, $D_2$, and $D_3$, grouping respectively elements 10–16, 25–32, and 44–50, respectively.

Step 2) For each of these three domains, all of the linearly independent selective vectors satisfying Eq. (4) are calculated and stored in the matrices $\tilde{\Theta}^j$ for $j = 1$–3. Respectively, 14, 16, and 14 selective vectors have been found, which are grouped in the single matrix $\tilde{\Theta} \in R^{100\times 44}$.

Step 3) All the linearly independent deselective vectors to these domains are similarly calculated from Eq. (5) and stored in the matrix $\tilde{\Psi} \in R^{100\times 56}$.

Step 4) Hence the initial basis of the model reduction is $\tilde{P} = [\tilde{\Theta} \mid \tilde{\Psi}]$, as in Eq. (6).

Step 5) We then search for the projection coefficient matrices $\tilde{C}_{\Theta^j}$ and $\tilde{C}_{\Psi}$ of the first 10 eigenvectors of the refined model, grouped in $Y^{(m)}$, respectively, on each of these bases $\tilde{\Theta}^j$ and on $\tilde{\Psi}$.

Step 6) A singular value decomposition is applied on each product $\tilde{\Theta}^j \tilde{C}_{\Theta^j}$. Figures 2 give, in each case, the singular values that we see decrease rapidly with mode number. From these curves, we retain 4, 5, and 3 vectors, respectively, for the domains $D_1$, $D_2$, and $D_3$ and store them in the matrix $\Theta \in R^{100\times 12}$. As well, a singular value decomposition is performed on $\tilde{\Psi}\tilde{C}_{\Psi}$. The 10 vectors retained are stored in $\Psi \in R^{100\times 10}$.

Step 7) We can now define the projection matrix in the model reduction $P = [\Theta \mid \Psi] \in R^{100\times 22}$.

Step 8) The reduced model of size $22 \times 22$ is finally obtained by projecting the stiffness and mass matrices of the refined model $K$, $M \in R^{100\times 100}$ on the basis $P$.

The block diagonal structure of the reduced-model stiffness matrix $\tilde{K}$ is visualized in Fig. 3. After the singular value decomposition, step 5, we retained 4, 5, and 3 vectors representing domains $D_1$, $D_2$, and $D_3$, respectively. These domains are decoupled in the reduced model, as seen from the nonoverlapping blocks in Fig. 3.

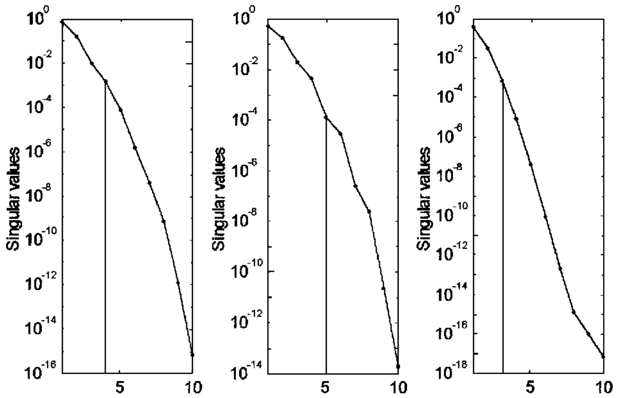



**Fig. 2 Singular value decompositions.**

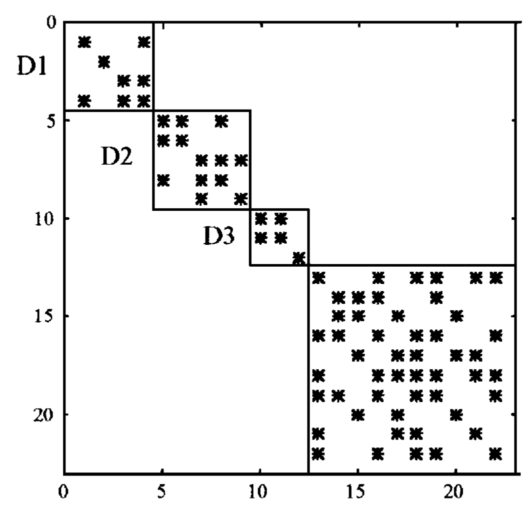



**Fig. 3 Visualization of block structure of reduced stiffness matrix.**

The model order reduction was implemented to preserve the first 10 modes of the refined model, as shown in Fig. 4. The left-hand plot of Fig. 4 represents the errors between the first 22 eigenfrequencies of the refined model and those of the reduced model, evaluated from the criterion

$$\varepsilon_\lambda(\%) = 100 \times \frac{|\lambda_v - \tilde{\lambda}_v|}{|\lambda_v|} \tag{10}$$

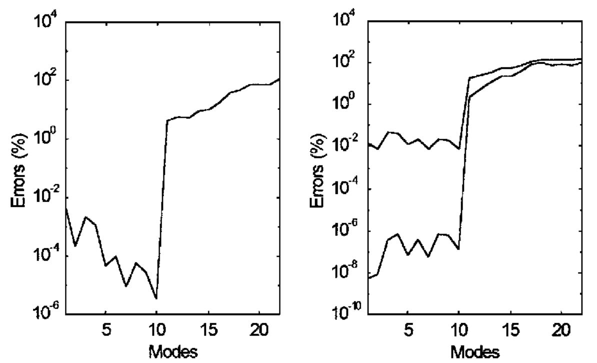
where the eigenvalues of the refined and reduced models are $\lambda_v$ and $\tilde{\lambda}_v$, respectively. We note that the spectrum of the refined model including the first 10 eigenfrequencies is preserved in the reduced model with an accuracy better than 0.01%. The right-hand plot in Fig. 4 gives the errors between the eigenvectors of both models. An expansion on the reduced-model eigenvectors has been performed to compare them with those of the refined model. The errors are calculated using two distance criteria, one based on the Modal Assurance Criteria (MAC) (solid line) and the other based on the relative error (dashed line):

$$\varepsilon_{\mathrm{MAC}} = 100 \times \left(1 - \frac{(y_v^T P z_v)^2}{\|y_v\|^2\,\|z_v\|^2}\right) \tag{11}$$

$$\varepsilon_{\mathrm{rel\,error}} = 100 \times \frac{\|y_v - P z_v\|}{\|y_v\|} \tag{12}$$

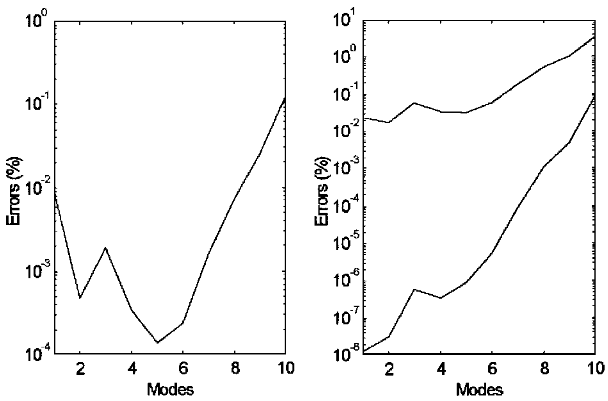
where the eigenvectors of the refined and the reduced models are $y_v$ and $z_v$, respectively. Note that even for the strictest criterion (relative-error criterion), the first 10 eigenvectors are preserved with between 0.01 and 0.1% error.

To illustrate the predictive capabilities of this reduced model we have modified the stiffnesses of domains $D_1$ and $D_2$ by factors of 0.5 and 1.5, respectively. The corresponding stiffnesses of the refined model have been likewise modified. We now calculate the eigenvalues of the reduced and refined models, and compare them according to Eq. (10) in the left-hand part of Fig. 5. We see that the




**Errors on eigenfrequencies** **Errors on eigenvectors**

**Fig. 4 Comparison between the refined and the reduced initial model behaviors.**




**Errors on eigenfrequencies** **Errors on eigenvectors**

**Fig. 5 Comparison between the refined and the reduced perturbed model behaviors.**

discrepancies between the reduced and refined eigenvalues are less than 0.1%, showing that the modifications of the refined model are accurately predicted from the modified reduced model. A similar test is made with the eigenvectors. The eigenvectors of the reduced and refined models after stiffness modification are compared in the right-hand part of Fig. 5, based on Eqs. (11) and (12). One sees that the eigenvectors of the modified refined model are reproduced by the modified reduced model with an error between 0.01 and 3% with the relative-error criterion, which is much stricter than the MAC criterion.

## V. Discussion

Model reduction procedures based on Eq. (3) share the same strength and weakness, namely, their performance intimately depends on the degree to which the column space of $P$ spans the desired response space of the refined model. On the one hand, this property can be used to construct very significant order reductions while retaining high accuracy. On the other hand, a poorly selected Ritz basis, even containing a relatively large number of vectors, inevitably will lead to disappointing results.

The model reduction algorithm presented above is based on a transformation $P$ that preserves certain topological properties of the refined model. The existence of these selective and deselective vectors depends exclusively on the connectivity of the model in question and is independent of the nominal values of the design parameters. For subdomains having complex connectivities and boundary conditions, the corresponding selective and deselective vector spaces may be reduced to very small dimensions and may even be empty sets, in which case the proposed algorithm will not be applicable. However, a large number of technological structures possess a relatively serial topology (rotors, space structures, etc.) and can be advantageously represented by reduced models obtained by the proposed procedure.

## VI. Conclusions

A model order reduction strategy has been developed on the basis of the concept of selective sensitivity. This model reduction has the property of preserving selected global behaviors of the refined model while reducing its dimension. The distinctive feature of this model order reduction lies in the property of topological localization, wherein subsets of refined model degrees of freedom can be mapped to subsets of reduced-model degrees of freedom. We have demonstrated that this property is the direct consequence of the selective and deselective properties of the projection matrix $P$.

## Appendix: Topological Localization Theorem and Related Results

Consider a conservative linear elastic system whose $N$-dimensional refined model is characterized by the stiffness and mass matrices $K$, $M$. The $N \times N$ dynamic stiffness matrix is given by Eq. (1). We assume that $Z(\lambda)$ has the submodeling property of Eq. (2).

Let us now define a model order reduction based on a linear transformation between the refined- and the reduced-model displacements $y$ and $z$, respectively:

$$y = Pz \tag{A1}$$

The stiffness and mass matrices of the reduced model are given by the relations of Eq. (3), where $K$ and $M$ are, respectively, the stiffness and mass matrices of the refined model and $P$ is a given projection matrix.

Given some general properties of the projection matrix $P$, the model reduction defined by Eq. (3) can have some basic properties, which are presented in propositions 2 and 3. But first, some main definitions must be given.

*Definition 1:* Let $K$ be a stiffness matrix having the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$. A matrix $\Theta$ has the property of selectivity with respect to $K$ if there exists a collection of nonempty disjoint index sets $D_1, \ldots, D_p$, none containing all $q$ indices, and if there exist $p$ submatrices $\Theta^j$ of $\Theta = [\Theta^1, \ldots, \Theta^p]$, $\Theta^j \in R^{N \times d_j}$, such that for each column $\theta \in \Theta^j$,

$$\sum_{i \in D_j} K_i \theta \neq 0 \quad \text{and} \quad \sum_{i \notin D_j} K_i \theta = 0 \qquad j = 1, \ldots, p \tag{A2}$$

*Definition 2:* Let $K$ be a stiffness matrix having the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$. A matrix $\Psi$ has the property of deselectivity with respect to $K$ for the index sets $D_1, \ldots, D_p$ if, for each column $\psi \in \Psi$,

$$\sum_{i \in D} K_i \psi = 0 \quad \text{and} \quad \sum_{i \notin D} K_i \psi \neq 0 \tag{A3}$$

where $D$ represents the union of the index sets $D_j$ ($j = 1, \ldots, p$), $D = \bigcup_j D_j$.

*Definition 3:* Given two matrices of the same dimension $A$ and $B$. The matrix $A \odot B$ is formed as the term-by-term product of $A$ and $B$:

$$[A \odot B]_{ij} = A_{ij} B_{ij} \tag{A4}$$

*Definition 4:* Let $\tilde{K} = P^T K P$ represent a model order reduction where the refined stiffness matrix $K$ has the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$. This model order reduction has the property of topological localization if there exists a collection of nonempty disjoint index sets $D_1, \ldots, D_p$, none containing all $q$ indices, such that the derivatives of $\tilde{K}$ satisfy

$$\left(\sum_{i \in D_m} \frac{\partial \tilde{K}}{\partial k_i}\right) \odot \left(\sum_{i \in D_n} \frac{\partial \tilde{K}}{\partial k_i}\right) = 0$$

$$\text{for all} \quad m, n = 1, \ldots, p; \quad m \neq n \tag{A5}$$

Equation (A5) means that the variations of the reduced stiffness matrix with respect to the refined-model stiffness parameters, indexed respectively in disjoint sets $D_m$ and $D_n$ ($m \neq n$), are independent in the sense of being topologically nonoverlapping. That is, variation of a stiffness parameter $k_i$ in the refined model causes variation of parameters in the reduced model that are unaffected by any other refined parameter. This establishes a unique correspondence between the physical model parameters of the reduced and refined models. This is important in situations where analyses performed on the reduced model are used to suggest physical properties or design changes for the refined model.

*Proposition 1:* Let $\tilde{K} = P^T K P$ be a model order reduction where the refined stiffness matrix $K$ has the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$ and where $P = [\Theta \mid \Psi]$, $\Theta$ having the property of selectivity with respect to $K$ and $\Psi$ having the property of deselectivity with respect to $K$ with index sets $D_1, \ldots, D_p$. The reduced stiffness matrix $\tilde{K}$ is a block-diagonal matrix:

$$\tilde{K} = \text{bdiag}(\Theta^T K \Theta, \Psi^T K \Psi) \tag{A6}$$

where bdiag( ) is the block-diagonal operator.

*Proof:* The reduced stiffness matrix $\tilde{K}$ is given by

$$\tilde{K} = P^T K P \tag{A7}$$

Considering $P = [\Theta \mid \Psi]$, Eq. (A7) becomes

$$\tilde{K} = \begin{bmatrix} \Theta^T K \Theta & \Theta^T K \Psi \\ \Psi^T K \Theta & \Psi^T K \Psi \end{bmatrix} \tag{A8}$$

Let $\theta \in \Theta^j$ ($j = 1, \ldots, p$) and $\psi \in \Psi$; then we have

$$\psi^T K \theta = \psi^T \left( \sum_{i \in D_j} k_i K_i + \sum_{i \notin D_j} k_i K_i \right) \theta \tag{A9}$$

Equations (A2) and (A3) imply that $\psi^T K \theta = 0$. Furthermore, $\psi^T K \theta = \theta^T K \psi$, and so, we have $\theta^T K \psi = 0$. This implies that the submatrices $\Psi^T K \Theta$ and $\Theta^T K \Psi$ are null matrices, demonstrating the block diagonality of the matrix $\tilde{K}$. □

Proposition 2 shows a necessary and sufficient condition for the first block of the reduced model $\tilde{K}_1 = \Theta^T K \Theta$ to have the property of topological localization.

*Proposition 2:* Let $\tilde{K} = P^T K P$ be a model order reduction where the refined stiffness matrix $K$ has the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$ and where $P = [\Theta \mid \Psi]$, $\Theta$ having the property of selectivity with respect to $K$ and $\Psi$ having the property of deselectivity with respect to $K$ with index sets $D_1, \ldots, D_p$. The property of topological localization is satisfied by the block $\tilde{K}_1 = \Theta^T K \Theta$ if and only if the projection submatrix $\Theta$ has the property of selectivity.

*Proof:*

1) We first demonstrate that the selectivity property is sufficient for topological localization. Let $\tilde{K}_1 = \Theta^T K \Theta$ be part of the reduced model $\tilde{K} = P^T K P$, where $\Theta = [\Theta^1, \ldots, \Theta^p]$, with $\Theta^j \in R^{N \times d_j}$, has the property of selectivity with respect to $K$:

$$\sum_{i \in D_j} K_i \theta \neq 0 \quad \text{and} \quad \sum_{i \notin D_j} K_i \theta = 0$$
$$\text{for all} \quad j = 1, \ldots, p, \quad \text{for all} \quad \theta \in \Theta^j \tag{A10}$$

We show that $\tilde{K}_1$ has the property of topological localization.

For any index set $D_j$, $j = 1, \ldots, p$, the derivatives of $\tilde{K}_1$ are given by

$$\sum_{i \in D_j} \frac{\partial \tilde{K}_1}{\partial k_i} = \sum_{i \in D_j} \frac{\partial}{\partial k_i} (\Theta^T K \Theta) \tag{A11}$$

$$= \Theta^T \sum_{i \in D_j} \frac{\partial}{\partial k_i} \left( \sum_{l=1}^{q} k_l K_l \right) \Theta \tag{A12}$$

$$= \Theta^T \sum_{i \in D_j} K_i \Theta \tag{A13}$$

To simplify the notations, define

$$B^m = \Theta^T \sum_{i \in D_m} K_i \Theta \quad \text{and} \quad \sigma_m = \sum_{l=1}^{m} d_l$$

where $\sigma_m$ is a column index locating the last vector of the submatrix $\Theta^m$, $\Theta^m \in R^{N \times d_m}$, in the matrix $\Theta$.

Because of the selective property of $\Theta$, we have the following property of the $(ij)$th element of $B^m$:

$$B^m_{ij} = 0 \quad \text{for all} \quad m = 1, \ldots, p$$
$$\text{if} \quad i \text{ or } j \leq \sigma_{m-1} \quad \text{or} \quad i \text{ or } j > \sigma_m \tag{A14}$$

If we can show that, for $m, n = 1, \ldots, p$ with $m \neq n$, $B^m_{ij} B^n_{ij} = 0$ for all $i \neq j$, then we have demonstrated that $\tilde{K}_1$ satisfies the property of topological localization.

Suppose there exists $i, j$ such that $B^m_{ij}\ B^n_{ij} \neq 0$ $m, n = 1, \ldots, p$; $m \neq n$. This implies that $B^m_{ij}$ and $B^n_{ij}$ both are nonzero terms: $B^m_{ij} \neq 0$ and $B^n_{ij} \neq 0$. According to Eq. (A14), these conditions are satisfied for $i, j$ such that

$$\sigma_{m-1} < i \text{ and } j \leq \sigma_m \quad \text{and} \quad \sigma_{n-1} < i \text{ and } j \leq \sigma_n \tag{A15}$$

Without loss of generality, we can choose $m < n$; we have $\sigma_m \leq \sigma_{n-1}$, and so, $i, j \leq \sigma_{n-1}$ and $i, j > \sigma_{n-1}$, which is a contradiction. Then for all $m, n = 1, \ldots, p$; $m \neq n$:

$$B^m_{ij} B^n_{ij} = 0 \quad \text{for all} \quad i, j \tag{A16}$$

which is equivalent to

$$B^m \odot B^n = 0 \tag{A17}$$

We have thus proven that selective sensitivity implies topological localization.

2) We now demonstrate that the selectivity property of $\Theta$ is a necessary condition for topological localization. Let $\tilde{K}_1 = \Theta^T K \Theta$ be part of the reduced model $\tilde{K}$, where $\tilde{K}_1$ has the property of topological localization:

$$\left( \sum_{i \in D_m} \frac{\partial \tilde{K}_1}{\partial k_i} \right) \odot \left( \sum_{i \in D_n} \frac{\partial \tilde{K}_1}{\partial k_i} \right) = 0$$
$$\text{for all} \quad m, n = 1, \ldots, p; \quad m \neq n \tag{A18}$$

We will show that $\Theta$ has the property of selectivity with respect to $K$.

By definition of the matrices $B^m$, Eq. (A18) is satisfied if and only if $B^m_{ij} B^n_{ij} = 0$ for all $i, j$. In particular, this has to be true for $i = j$:

$$B^m_{ii} B^n_{ii} = 0 \tag{A19}$$

Eq. (A19) is satisfied by any of the three following configurations:

$$\begin{aligned} B^m_{ii} &= 0 & B^n_{ii} &\neq 0 \\ B^m_{ii} &\neq 0 & B^n_{ii} &= 0 \\ B^m_{ii} &= 0 & B^n_{ii} &= 0 \end{aligned} \tag{A20}$$

Suppose that $B^m_{ii} \neq 0$, which means that there exists $\theta \in \Theta$:

$$\theta^T \sum_{i \in D_m} K_i \theta \neq 0$$

Then, from Eq. (A20), we necessarily have $B^n_{ii} = 0$, that is,

$$\theta^T \sum_{i \in D_n} K_i \theta = 0$$

which implies

$$\sum_{i \in D_n} K_i \theta = 0$$

because

$$\theta^T \sum_{i \in D_n} K_i \theta$$

is a quadratic form where

$$\sum_{i \in D_n} K_i$$

is a symmetric and nonnegative matrix. In other words, considering a vector $\theta \in \Theta$ not orthogonal to the columns of

$$\sum_{i \in D_m} K_i$$

we have shown that $\theta$ is orthogonal to the columns of

$$\sum_{i \in D_n} K_i \ (m \neq n)$$

which is the definition of selectivity of a matrix. We thus have proven that topological localization implies selective sensitivity. □

This property of topological localization means that if one parameter of the refined model is modified, then the physical parameters of the reduced model that are affected by this change are easily localized.

A direct consequence of proposition 2 concerns the relationship existing between parameters of the refined model and the static responses of the reduced model. This relation is formulated in proposition 3.

*Proposition 3:* Let $\tilde{K} = P^T K P$ be a model order reduction, where the refined stiffness matrix $K$ has the submodeling property of Eq. (2) with $q$ submodels $K_1, \ldots, K_q$ and where $P = [\Theta \mid \Psi]$, $\Theta$ having the property of selectivity with respect to $K$ and $\Psi$ having the property of deselectivity with respect to $K$ with index sets $D_1, \ldots, D_p$. We assume that each set $D_j$ $(j = 1, \ldots, p)$ corresponds to a single, perhaps homogenized, stiffness parameter, and that the static response $y_s$ of the refined model due to an excitation $f$ can be represented on $P$. Then this static response $y_s$ is related to the stiffness parameters $k_j$ $(j = 1, \ldots, p)$ through the equation

$$y_s = y^S_s(k_j, f) + y^D_s(f) \tag{A21}$$

with

$$y^S_s(k_j, f) = \sum_{j=1}^{p} \frac{1}{k_j} \left[ \Theta^j \left( \Theta^{j^T} \sum_{i \in D_j} K_i \Theta^j \right)^{-1} \Theta^{j^T} f \right] \tag{A22}$$

and

$$y_s^D(f) = \Psi\left(\Psi^T \sum_{i \notin D_j, j=1,\ldots,p} k_i K_i \Psi\right)^{-1} \Psi^T f \tag{A23}$$

where $y_s^S(k_j, f)$ represents the component of the static displacement that is an explicit and simple function of the stiffness parameters $k_j$ $(j = 1, \ldots, p)$ and $y_s^D(f)$ represents the component of the static displacement that is independent of the aforementioned stiffness parameters.

*Proof:* The static response $y_s$ of the refined model to an excitation $f$ satisfies

$$Ky_s = f \tag{A24}$$

If we assume that $y_s$ can be represented on $P$, that is,

$$y_s = Pz_s \tag{A25}$$

then by premultiplying Eq. (A24) by $P^T$, we obtain

$$P^T KPz_s = P^T f \tag{A26}$$

or

$$\tilde{K}z_s = P^T f \tag{A27}$$

where $P = [\Theta^1 \ldots \Theta^p | \Psi]$, each $\Theta^j (j = 1, \ldots, p)$ containing linearly independent selective vectors to $D_j$ and $\Psi$ containing linearly independent deselective vectors to the union of all the index sets $D_j$ $(j = 1, \ldots, p)$, and $\tilde{K} = \text{bdiag}(\Theta^{1^T} K\Theta^1, \ldots, \Theta^{p^T} K\Theta^p, \Psi^T K\Psi)$.

$\tilde{K}$ is a nonsingular matrix because each of its blocks is a full rank matrix. Then, Eq. (A27) can be written

$$z_s = \tilde{K}^{-1} P^T f \tag{A28}$$

where

$$\tilde{K}^{-1} = \text{bdiag}\left[\frac{1}{k_1}\left(\Theta^{1^T} \sum_{i \in D_1} K_i \Theta^1\right)^{-1}, \ldots, \frac{1}{k_p}\left(\Theta^{p^T} \sum_{i \in D_p} K_i \Theta^p\right)^{-1}, (\Psi^T K \Psi)^{-1}\right]$$

because of the selective properties of $\Theta$ and the deselective properties of $\Psi$ with respect to $K$.

Taking into account Eqs. (A25) and (A28), we finally obtain relations (A21–A23), which explicitly relate the static response of the refined model, due to an excitation $f$, to the stiffness parameters $k_j$ $(j = 1, \ldots, p)$. □

The parameter property presented in proposition 3 is a consequence of the topological localization property, which is specific to model order reduction by selective sensitivity.

The two preceding propositions have established specific properties of model order reduction by selective sensitivity. The following ones are more general and concern any reduction strategy based on a full-rank linear transformation between the refined- and the reduced-model displacements, as defined in Eqs. (A1) and (3).

Proposition 4 shows in what sense the strain and kinetic energies of the refined model are preserved by the reduced model.

*Proposition 4:* Given the model order reduction defined in Eq. (3), let $y$ be an $N$-dimensional displacement vector of the refined model. If it can be exactly represented on the basis $P$, then the strain and kinetic energies of the refined model for this displacement are preserved in the reduced model.

*Proof:* The proposition assumes that there exists a vector $z$ such that $y = Pz$. The strain and kinetic energies of the refined model for the response $y$, respectively, given by $y^T Ky$ and $y^T My$, also can be written

$$y^T Ky = z^T P^T KPz = z^T \tilde{K}z \tag{A29}$$

$$y^T My = z^T P^T MPz = z^T \tilde{M}z \tag{A30}$$

where $z^T \tilde{K}z$ and $z^T \tilde{M}z$ are, respectively, the strain and kinetic energies of the reduced model for the response $z$, where $z$ is the response in the reduced model that is equivalent to the refined response $y$. □

Proposition 5 shows that a certain class of static displacement vectors is preserved by the model reduction.

*Proposition 5:* Given the model order reduction defined in Eq. (3), let $y_s$ be an $N$-dimensional static displacement vector of the refined model subject to the force $f$. If $y_s$ can be exactly represented on the basis $P$, then the static behavior of the refined model is preserved in the reduced model.

*Proof:* Let $f$ be the force that, when applied to the refined model, leads to the displacement $y_s$. It satisfies

$$Ky_s = f \tag{A31}$$

Premultiplying by $P^T$ and assuming that $y_s = Pz_s$, Eq. (A31) becomes

$$P^T KPz_s = P^T f \tag{A32}$$

or, using Eq. (3),

$$\tilde{K}z_s = P^T f \tag{A33}$$

where $z_s$ is the response of the reduced model to the equivalent force $P^T f$. □

Proposition 6 shows that part of the spectrum of the refined model is retained in the reduced model.

*Proposition 6:* Given the model order reduction defined in Eq. (3), let $y_\nu$ be an $N$-dimensional eigenvector of the refined model. If it can be represented exactly on $P$, then its associated eigenvalue is preserved in the spectrum of the reduced model and its projection is an eigenvector of the reduced model.

*Proof:* Let $y_\nu$ be an eigenvector of the refined model that satisfies

$$(K - \lambda_\nu M)y_\nu = 0 \tag{A34}$$

and assume that $y_\nu = Pc_\nu$. Then Eq. (A34) is written

$$(K - \lambda_\nu M)Pc_\nu = 0 \tag{A35}$$

and by premultiplying by $P^T$,

$$(P^T KP - \lambda_\nu P^T MP)c_\nu = 0 \tag{A36}$$

Considering Eq. (3), this becomes

$$(\tilde{K} - \lambda_\nu \tilde{M})c_\nu = 0 \tag{A37}$$

The eigenvalue $\lambda_\nu$ is preserved and the projection of $y_\nu$ on $P$ is the corresponding eigenvector of the reduced model, associated with $\lambda_\nu$. □

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## References

[1] Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[2] Craig, R. R., and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analyses," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1313–1319.

[3] MacNeal, R. H., "A Hybrid Method of Component Mode Synthesis," *Computers and Structures*, Vol. 1, 1971, pp. 581–601.

[4] Rubin, S., "Improved Component Mode Representation for Structural Dynamic Analysis," *AIAA Journal*, Vol. 13, No. 2, 1996, pp. 79–81.

[5] Ben-Haim, Y., "Adaptive Diagnosis of Faults in Elastic Structures by Static Displacement Measurement: The Method of Selective Sensitivity," *Mechanical Systems and Signal Processing*, Vol. 6, No. 1, 1992, pp. 85–96.

[6] Ben-Haim, Y., and Prells, U., "Selective Sensitivity in the Frequency Domain, Part I: Theory," *Mechanical Systems and Signal Processing*, Vol. 7, No. 5, 1993, pp. 461–475.

[7] Cogan, S., Lallement, G., Ayer, F., and Ben-Haim, Y., "Updating Linear Elastic Models with Modal Selective Sensitivity," *Inverse Problems in Engineering*, Vol. 2, 1995, pp. 29–47.

[8] Prells, U., and Ben-Haim, Y., "Selective Sensitivity in the Frequency Domain, Part II: Applications," *Mechanical Systems and Signal Processing*, Vol. 7, No. 6, 1993, pp. 551–574.

[9] Golub, G. H., and Van Loan, C. F., *Matrix Computations*, North Oxford Academic, London, 1986, pp. 18–20.

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